Xbox Xponential

Lesson by Mathalicious;
Annotation by Student Achievement Partners

GRADE LEVEL High School

IN THE STANDARDS F-IF.C.8b, F-BF.A.1a, F-LE.A.2, F-LE.B.5, S-ID.B.6a

WHAT WE LIKE ABOUT THIS LESSON

Mathematically:

- Requires students to interpret key features of an exponential function through a real-world model
- Connects an exponential function with its graph, table, and matching verbal description
- Includes several stages of the basic modeling cycle (computing, interpreting, and validating)

In the classroom:

- Captures student attention by using an engaging context
- Allows for student collaboration and discussion of mathematical concepts; includes guiding questions for teachers to use to facilitate discussion
- Uses technology to illustrate the mathematics of the lesson
- Provides ample time for students to engage deeply with the mathematics

MAKING THE SHIFTS[1]

Focus	Belongs to the Widely Applicable Prerequisites for College and Careers[2]
Coherence	Builds on the idea that exponential functions have a structurally similar growth principle to linear functions
Rigor[3]	Conceptual Understanding: secondary in this lesson
	Procedural Skill and Fluency: secondary in this lesson
	Application: primary in this lesson

[1]For more information read Shifts for Mathematics.
[2]For more information, see Widely Applicable Prerequisites for College and Careers.
[3]Lessons may target one or more aspect(s) of rigor.

ADDITIONAL THOUGHTS

This lesson is a continuation of students' learning about exponential functions, and is not intended to be an introduction to exponential functions. The lesson assumes, and builds on, students' understanding of exponents, square roots, and percentages. This particular lesson, likely to take several days for students to complete, focuses on an exponential growth model based on Moore's Law. It is not intended for students to meet the full expectations of the course-level standards through only this lesson.

This lesson begins with a focus on building a function (F-BF) to match a verbal description, followed by interpreting and comparing the function (F-IF) to real data. It concludes with building a new, improved function that more accurately models actual data. The premise of the lesson is entirely based on exponential models (F-LE). For more insight on the course-level concepts from the Functions conceptual category addressed in this lesson, read pages 7–17 of the progression document, Grade 8, High School, Functions.

Within this lesson, students apply key understandings of functions and equations from previous grades or courses. The "Preview & Guiding Questions" section offers students the opportunity to use their knowledge of dependent and independent variables, their relationships, and how graphs represent those relationships as a way to quickly engage with the task. Throughout the lesson, there are examples of scaffolded discussion questions that may help students think about the structure of the exponential function they are examining. For example, questions like those below can help make the mathematics of the lesson more explicit for students.

- What operation did you keep doing over and over again?
- Once you know the number of doublings, how can you find the processor speed?
- Does your equation agree with the predicted speeds in your table?
- Is the annual growth rate greater than or less than 50%? How do you know?

Teachers and those who support teachers may find the Instructional Practice Guide: Coaching tool useful in implementing best practices in the classroom that allow all students to master the content of the lesson.

XBOX XPONENTIAL

How have video game console speeds changed over time?

lesson guide

In 1965 Gordon Moore, computer scientist and Intel co-founder, predicted that computer processor speeds would double every two years. Twelve years later the first modern video game console, the Atari 2600, was released.

In this lesson, students write an exponential function based on the Atari 2600 and Moore's Law, and research other consoles to determine whether they've followed Moore's Law.

Primary Objectives

- Apply an exponential growth model, stated verbally, to various inputs
- Generalize with an exponential function to model processor speed for a given year
- Research actual processor speeds, and compare them to the model's prediction
- Calculate the *annual* growth rate of the model (given biannual growth rate)
- Use technology to model the actual processor speeds with an exponential function
- Interpret the components of the regression function in this context, and compare them to the model

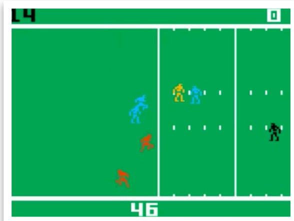
Content Standards (CCSS)	Mathematical Practices (CCMP)	Materials
Functions: IF.8b, BF.1a, LE.2, LE.5 Statistics: ID.6a	MP.4, MP.7	• Student handout • LCD projector • Computer speakers • Graphing calculators • Computers with Internet access

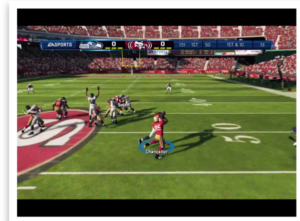
Before Beginning…

Students should be familiar with the meaning of and notation for exponents, square roots, percent growth and the basics of exponential functions of the general form $y = ab^x$. Students will need to enter data in calculator lists and perform an exponential regression, so if they're inexperienced with this process, you will need time to demonstrate.

Preview & Guiding Questions

We'll begin by watching a short video showing the evolution of football video games.





Ask students to sketch a rough graph of how football games have changed over time. Some will come up with a graph that increases linearly, perhaps some increasing at an accelerating rate. Some students may show great leaps in technology with new inventions, while others may show the quality leveling off in the more recent past.

Then, ask them to label the axes. The horizontal axis will be time in years, but what about the vertical axis? Ask students to describe what they are measuring, exactly, when they express the quality of a video game. They might suggest realism, speed or power. Students should try to explain how they would measure these (or others they come up with), and realize that while a subjective element like "realism" is difficult to quantify, it is possible to measure speed (in MHz) of a console's processor.

- *Sketch a graph of how you think video games have changed over time.*
- *What was the reasoning behind the shape of the graph you sketched?*
- *What does your horizontal axis represent?*
- *What label did you assign to the vertical axis? Which of these are measureable?*

Act One

In 1965 Gordon Moore, computer scientist and Intel co-founder, predicted that computer processor speeds would double every two years. Starting with the 1.2 MHz Atari 2600 in 1977 (the first console with an internal microprocessor), students apply the rule "doubles every two years" to predict the speed of consoles released in several different years. By extending the rule far into the future, they are motivated to write a function to model processor speed in terms of release year: $1.2 \cdot 2^{t/2}$. They will understand that 1.2 represents the speed of the initial processor, the base of 2 is due to doubling, and the exponent $t/2$ represents the number of doublings.

Act Two

How does the prediction compare to what has actually happened? Students research the actual processor speed of several consoles released over the years. By comparing predicted vs. actual processor speeds in a table, we see that they were slower than Moore's Law predicted. How different are the models, though? Students first algebraically manipulate the "doubling every two years" model to create one that expresses the growth rate each year. Then, they use the list and regression functionality of their graphing calculators to create an exponential function that models the actual data. By comparing the two functions, they conclude that while the actual annual growth rate (30%) was slower than the predicted annual growth rate based on Moore's Law (41%), the Atari 2600 was also ahead of its time.

Act One: Moore Fast

1 In 1965, computer scientist Gordon Moore predicted that computer processor speeds would double every two years. Twelve years later, Atari released the 2600 with a processor speed of 1.2 MHz.

Based on **Moore's Law**, how fast would you expect the processors to be in each of the consoles below?

 Atari 2600 1977	 Intellivision 1979	 N.E.S. 1983	 Atari Jaguar 1993	 GameCube 2001	 XBOX 360 2005
1.2 MHz	*2.4 MHz*	*9.6 MHz*	*307.2 MHz*	*4,915 MHz*	*19,661 MHz*

×2 *×2×2* *×2×2×2×2×2* *×2×2×2×2* *×2×2*

Explanation & Guiding Questions

Before turning students loose on this question, make sure they can articulate the rule "doubles every two years".

It is common for students to correctly double 1.2MHz and get 2.4 MHz in 1979, but then to continue adding 1.2 at a constant rate every two years. Most will self-correct as they check in with their neighbors, but be on the lookout for that misunderstanding of the pattern.

Once students have finished the table, and some have started to think about the next question, you can display the answers and prompt students to explain their reasoning.

- *Restate Moore's Law in your own words.*
- *How many times should the processor speed have doubled between the release of the Intellivision and the release of the N.E.S.?*
- *What operation did you keep doing over and over again?*
- *Where did that 307.2 come from? How did you calculate that?*

Deeper Understanding

- *What's an easier way to write ×2×2×2×2×2? (×2^5)*
- *In what year would Gordon Moore say a 76.8 MHz processor would be released? (1989, since 76.8 = 9.6 × 2^3, so 6 years after 1983.)*

2 Assuming Moore's Law is true, write an expression to estimate how fast console processors should be in 2077, a century after the original Atari.

$1.2 \cdot 2^{50}$

Explanation & Guiding Questions

The year 2077 is exactly 100 years after the release of the Atari 2600. Since Moore's Law predicts that processing speeds will double every *two* years, that means that the Atari 2600's processing speed should have doubled *fifty* times. If students have trouble discerning the pattern, try adding a row to the table in the previous question:

Atari 2600	Intellivision	N.E.S.	Atari Jaguar	GameCube	XBOX 360
0 years	2 years	6 years	16 years	24 years	28 years
0 doublings	1 doubling	3 doublings	8 doublings	12 doublings	14 doublings
$1.2 \cdot 2^0$	$1.2 \cdot 2^1$	$1.2 \cdot 2^3$	$1.2 \cdot 2^8$	$1.2 \cdot 2^{12}$	$1.2 \cdot 2^{14}$

The key realization is that, when writing an expression, we don't really care about the number of *years* that have passed as much as the number of *doublings* that have occurred since the Atari was released. Since processor speeds double every two years, the number of doublings is the number of years divided by two. If students recognize this, they'll have a much easier time writing the equation for the Moore's Law processor speed in the next question.

- *How many doublings would occur after 8 years? 12 years? 100 years?*
- *Once you know the number of doublings, how can you find the processor speed?*

Deeper Understanding

- *If the processor speed doubled 8 times, how many years is that? (16 years, because it doubles every 2 years.)*
- *Is $1.2 \cdot 2^{50}$ MHz reasonable? How big is that? (On the order of 10^{15}, or 10^{12} GHz. It would be bananas.)*

3 We can think of the release of the Atari 2600 as the start of the video game era: 1977 CE becomes year 0 VG. Let the variable *t* represent the number of years that have passed since 1977, i.e. the video game year. Write an equation to calculate the expected processor speed for a given video game year.

$$speed = 1.2 \cdot 2^{\frac{t}{2}}$$

Explanation & Guiding Questions

By this point, students have already recognized the exponential growth pattern and written one expression to calculate processing speed in a given year. The main difficulty in generalizing this pattern for an arbitrary year is that the exponent doubles once every *two* years. One strategy to help students figure out the correct form of the exponent is to first write the equation in terms of the *number of doublings*, and then adjust the exponent accordingly. For instance:

step i. Speed = $1.2 \cdot 2^d$, where d = the number of doublings

step ii. $d = \frac{t}{2}$, since the number of doublings is half the number of years that have passed since 1977

step iii. Therefore, substituting for d gives $1.2 \cdot 2^{\frac{t}{2}}$

Act One has been all about Moore's Law, which was a prediction. It is natural to wonder about what actually happened. Was Gordon Moore right? How would we know? If he was not exactly correct, how close was he?

- *What must the expected processor speed be in 0 VG? 2 VG? 8 VG?*
- *When we wanted to make a prediction for 100 years, what exponent did we use?*
- *When you know how many years, how many doublings is that?*
- *Does your equation agree with the predicted speeds in your table?*
- *This equation allows us to* ***predict*** *the processor speed for any year. What are you curious about now?*

Deeper Understanding

- *If you graph this with technology, what window settings would be reasonable? ($0 \le x \le 40$, $0 \le y \le 2^{20}$.)*
- *Can you rewrite the expression to give you the predicted speed, where the input was the calendar year? (It would be $speed = 1.2 \cdot 2^{\frac{c - 1977}{2}}$, where c = calendar year.)*
- *What would be some advantages of using this version instead? (We wouldn't have to figure out the VG year, the number of years after 1977, before using the expression.)*
- *What would be some disadvantages of using this version instead? (The y-intercept would be the processor speed in the year 0 CE. If we wanted to graph it, the graph would not be meaningful until c = 1977.)*

Act Two: Past, Present...

4 Using a site like Wikipedia, find out when each console below was released and calculate its predicted processor speed. Then find out the actual processor speeds. Have video game processor speeds followed Moore's Law? (Note: 1 GHz = 1000 MHz. Also, if a console uses multiple processors, use the fastest one.)

Console	Release	VG Year	Best selling game	MHz, Predicted	MHz, Actual
Atari 2600	1977	0	Pac-Man	1.2	1.2
Mattel Intellivision	*1979*	*2*	*Astrosmash*	*2.4*	*0.9*
Nintendo N.E.S.	*1983*	*6*	*Super Mario Brothers*	*9.6*	*1.8*
Sega Genesis	*1988*	*11*	*Sonic the Hedgehog 2*	*54.3*	*7.7*
Super N.E.S.	*1990*	*13*	*Super Mario World*	*108.6*	*3.6*
Atari Jaguar	*1991*	*14*	*Alien vs. Predator*	*153.6*	*26.6*
Sony PlayStation	*1994*	*17*	*Gran Turismo*	*434.4*	*33.9*
Nintendo 64	*1996*	*19*	*Super Mario 64*	*868.9*	*93.8*
Sony PlayStation 2	*2000*	*23*	*Grand Theft Auto*	*3,475.6*	*299*
Nintendo GameCube	*2001*	*24*	*Super Smash Bros. Melee*	*4,915.2*	*486*
Microsoft XBOX	*2001*	*24*	*Halo 2*	*4,915.2*	*733*
Microsoft XBOX 360	*2005*	*28*	*Kinect Adventures*	*19,660.8*	*3,200*
Sony PlayStation 3	*2006*	*29*	*Black Ops*	*27,804.6*	*3,200*
Nintendo Wii	*2006*	*29*	*Wii Sports*	*27,804.6*	*729*
Nintendo Wii U	*2012*	*35*	*Nintendo Land*	*222,436.6*	*1,240*

Explanation & Guiding Questions

This research will take some time. Groups of students could save time by dividing up the work. We recommend providing students access to laptops or tablets, instead of using smartphones. Wikipedia has a series of pages titled *History of Video Game Consoles* that include tables with side-by-side comparisons of release dates and processor speeds. The tables are much easier to read on a larger screen.

In the course of research, some questions will come up. For example, data for both the CPU (central processor) and GPU (graphics processor) are often reported. In this lesson, we are talking about CPU's. Additionally, release dates for different parts of the world might be available. It's easiest to look for the North American release date, since those are always reported. Using real data can be messy! Students will learn that statistical work often involves making decisions and drawing conclusions based on those decisions.

Before continuing, students should also complete the "predicted speed" column, based on the equation they wrote previously. (If they try to just double, they'll run into a problem with odd numbers in the domain.) The whole point of this exercise is to compare Moore's Law's prediction with what actually happened, with side-by-side results.

- *What do YOU think we should record (when there are multiple dates or speeds reported)?*
- *What observations can you make about the actual processor speeds?*
- *How good was Gordon Moore's prediction, applied to video game consoles?*

Deeper Understanding

- *What decisions did we have to make while collecting data? What are the implications? (Answers will vary.)*

5 As you know, the Moore's Law equation predicts that speeds would start at 1.2 MHz and double every *two* years. But how does this compare to how processor speeds have *actually* changed over time?

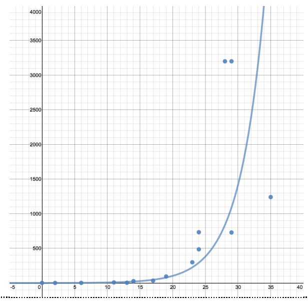
a. Using technology, write a function that fits the data as closely as possible.

speed = 0.54 · 1.3^t

or

speed = 0.54 · $e^{0.27t}$

(These are equivalent, since $e^{0.27} \approx 1.3$. The first is easier to interpret.)



b. Using your new equation, determine the actual initial starting speed and annual growth rate.

By interpreting the regression equation,

speed = 0.54 · 1.3^t

the initial processor speed is 0.54 MHz and the annual growth rate is 30%.

c. Calculate the *annual* growth rate predicted by Moore's Law.

The processor speed must grow by a factor of $\sqrt{2}$ each year (because $\sqrt{2} \cdot \sqrt{2} = 2$).

Since $\sqrt{2} \approx 1.41$, that means the annual growth rate is around 41%.

d. How reasonable was Moore's prediction for video game console processor speeds?

The predicted annual growth rate of 41% isn't that different from 30%. The Atari's initial processor speed was 1.2 MHz, suggesting it was ahead of its time. The model is imperfect, but pretty reasonable.

Explanation & Guiding Questions

How much support students will need in finding and interpreting a regression equation will depend on how much experience they've had with such a process.

Finding the annual growth rate within the Moore's Law prediction ($1.2 \cdot 2^{\frac{t}{2}}$) is the most difficult part of this question. Initially, many students will reason that the predicted annual growth rate must be 50%, since processing speed is growing by 100% every two years. Of course, if the speed grew 50% the first year, the total after two years would be *more than* 100% ($1.5^2 = 2.25$, so 125%) due to the compounding. If students have trouble reasoning about $\sqrt{2} \cdot \sqrt{2}$, exponent rules show that $1.2 \cdot 2^{\frac{t}{2}} = 1.2 \cdot \left(2^{\frac{1}{2}}\right)^t \approx 1.2 \cdot 1.41^t$.

Even though the actual processor speeds appear to be way below the predicted values at first glance, Moore's Law isn't such a bad fit after all. The growth rate is a little too high, but the exponential model is a reasonable one.

- *50% annual growth is a great guess. Can you convince me it's wrong?*
- *Is the annual growth rate greater than or less than 50%? How do you know?*
- *We are multiplying by something, and then by that same thing again, and the result is the same as if we had multiplied by 2... (if necessary: What times itself is 2?)*
- *What is a decimal approximation of $\sqrt{2}$? What percent growth rate corresponds to this?*
- *Which aspects of Gordon Moore's prediction were correct?*

Deeper Understanding

- *How might the model change if we took out Nintendo? (Nintendo's processors have been slower than the rest of the industry. Without their data, the model might match Moore's Law more closely.)*

XBOX XPONENTIAL

How have video game console speeds changed over time?

name

date

Act One: Moore Fast

1 In 1965, computer scientist Gordon Moore predicted that computer processor speeds would double every two years. Twelve years later, Atari released the 2600 with a processor speed of 1.2 MHz.

Based on **Moore's Law**, how fast would you expect the processors to be in each of the consoles below?

 Atari 2600 1977	 Intellivision 1979	 N.E.S. 1983	 Atari Jaguar 1993	 GameCube 2001	 XBOX 360 2005
1.2 MHz					

2 Assuming Moore's Law is true, write an expression to estimate how fast console processors should be in 2077, a century after the original Atari.

3 We can think of the release of the Atari 2600 as the start of the video game era: 1977 CE becomes year 0 VG. Let the variable *t* represent the number of years that have passed since 1977, i.e. the video game year. Write an equation to calculate the expected processor speed for a given video game year.

Act Two: Past, Present…

4 Using a site like Wikipedia, find out when each console below was released and calculate its predicted processor speed. Then find out the actual processor speeds. Have video game processor speeds followed Moore's Law? (Note: 1 GHz = 1000 MHz. Also, if a console uses multiple processors, use the fastest one.)

Console	Release	VG Year	Best selling game	MHz, Predicted	MHz, Actual
Atari 2600	1977	0	Pac-Man	1.2	1.2
Mattel Intellivision					
Nintendo N.E.S.					
Sega Genesis					
Super N.E.S.					
Atari Jaguar					
Sony PlayStation					
Nintendo 64					
Sony PlayStation 2					
Nintendo GameCube					
Microsoft XBOX					
Microsoft XBOX 360					
Sony PlayStation 3					
Nintendo Wii					
Nintendo Wii U					

5 As you know, the Moore's Law equation predicts that speeds would start at 1.2 MHz and double every *two* years. But how does this compare to how processor speeds have *actually* changed over time?

a. Using technology, write a function that fits the data as closely as possible.	b. Using your new equation, determine the actual initial starting speed and annual growth rate.
c. Calculate the *annual* growth rate predicted by Moore's Law.	d. How reasonable was Moore's prediction for video game console processor speeds?