Standards for Mathematical Practice: Commentary and Elaborations for K–5

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For discussion of the Elaborations and related topics, see the Tools for the Common Core blog: http://commoncoretools.me.

The Standards for Mathematical Practice, annotated for the K–5 classroom

The Common Core State Standards describe the Standards for Mathematical Practice this way:

> The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report Adding It Up: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

In this document we provide two different ways of adapting the language of the practice standards to the K–5 setting.

In this section we provide annotated versions of the standards that provide additional interpretation of the standards appropriate for K–5 classrooms. This section is intended for people who want to understand how the original language of the standards applies in K–5.

In the next section we provide *elaborations* of the standards: narrative descriptions that integrate the annotations from the first section and provide a coherent description of how the practice standards play out in the K–5 classroom.

1. Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution.• They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution.• They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem.• Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?"• They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

• Young students might use concrete objects or pictures to show the actions of a problem, such as counting out and joining two sets to solve an addition problem. If students are not at first making sense of a problem or seeing a way to begin, they ask questions that will help them get started.

• For example, to solve a problem involving multidigit numbers, they might first consider similar problems that involve multiples of ten or one hundred. Once they have a solution, they look back at the problem to determine if the solution is reasonable and accurate. They often check their answers to problems using a different method or approach.

• Mathematically proficient elementary students may consider different representations of the problem and different solution pathways, both their own and those of other students, in order to identify and analyze correspondences among approaches.

• When they find that their solution pathway does not make sense, they look for another pathway that does.

2. Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize•—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand;• considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

• For example, to find the area of the floor of a rectangular room that measures 10 m. by 12 m., a student might represent the problem as an equation, solve it mentally, and record the problem and solution as $10 \times 12 = 120$. He has decontextualized the problem. When he states at the end that the area of the room is 120 square meters, he has contextualized the answer in order to solve the original problem. Problems like this that begin with a context and are then represented with mathematical objects or symbols are also examples of modeling with mathematics (MP.4).

• For example, when a student sees the expression $40 - 26$, she might visualize this problem by thinking, if I have 26 marbles and Marie has 40, how many more do I need to have as many as Marie? Then, in that context, she thinks, 4 more will get me to a total of 30, and then 10 more will get me to 40, so the answer is 14. In this example, the student uses a context to think through a strategy for solving the problem, using the relationship between addition and subtraction and decomposing and recomposing the quantities. She then uses what she did in the context to identify the solution of the original abstract problem.

3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments.• They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples.• They justify their conclusions, communicate them to others, and respond to the arguments of others.• They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions.• Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies.• Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

• For example, a student might argue that two different shapes have equal area because it has already been demonstrated that both shapes are half of the same rectangle.

• For example, a rhombus is an example that shows that not all quadrilaterals with 4 equal sides are squares; or, multiplying by 1 shows that a product of two whole numbers is not always greater than each factor.

• Students present their arguments in the form of representations, actions on those representations, and explanations in words (oral or written). In the elementary grades, arguments are often a combination of all three. Some of their arguments apply to individual problems, but others are about conjectures based on regularities they have noticed across multiple problems (see MP.8, Look for and express regularity in repeated reasoning).

• For example, in order to demonstrate what happens to the sum when the same amount is added to one addend and subtracted from another, students in the early grades might represent a story about children moving between two classrooms: the number of children in each classroom is an addend; the total number of children in the two classrooms is the sum. When some students move from one classroom to the other, the number of students in each classroom changes by that amount—one addend decreases by some amount and the other addend increases by that same amount—but the total number of students does not change. An older elementary student might use an area representation to show why the distributive property holds.

• For example, young students may believe a generalization about the behavior of addition applies to positive whole numbers less than 100 because those are the numbers with which they are currently familiar. As they expand their understanding of the number system, they may reexamine their conjecture for numbers in the hundreds and thousands.

In upper elementary grades, students return to their conjectures and arguments about whole numbers to determine whether they apply to fractions and decimals. For example, students might make an argument based on an area representation of multiplication to show that the distributive property applies to problems involving fractions.

4. Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.• In early grades, this might be as simple as writing an addition equation to describe a situation.• In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas.• They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.•

• For elementary students, this includes the contextual situations they encounter in the classroom. When elementary students are first studying an operation such as addition, they might arrange counters to solve problems such as this one: there are seven animals in the yard, some are dogs and some are cats, how many of each could there be? They are using the counters to model the mathematical elements of the contextual problem—that they can split a set of 7 into a set of 3 and a set of 4. When they learn how to write their actions with the counters in an equation, $4 + 3 = 7$, they are modeling the situation with numbers and symbols. Similarly, when students encounter situations such as sharing a pan of cornbread among 6 people, they might first show how to divide the cornbread into 6 equal pieces using a picture of a rectangle. The rectangle divided into 6 equal pieces is a model of the essential mathematical elements of the situation. When the students learn to write the name of each piece in relation to the whole pan as 1/6, they are now modeling the situation with mathematical notation.

• In addition to numbers and symbols, elementary students might use geometric figures, pictures or physical objects used to abstract the mathematical elements of the situation, or a mathematical diagram such as a number line, a table, or a graph, or students might use more than one of these to help them interpret the situation.

• For example, if there is a Penny Jar that starts with 3 pennies in the jar, and 4 pennies are added each day, students might use a table to model the relationship between number of days and number of pennies in the jar. They can then use the model to determine how many pennies are in the jar after 10 days, which in turn helps them model the situation with the expression, $4 \times 10 + 3$.

• As students model situations with mathematics, they are choosing tools appropriately (MP.5). As they decontextualize the situation and represent it mathematically, they are also reasoning abstractly (MP.2).

5. Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software.• Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations.• For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.•

• The tools that elementary students might use include physical objects (cubes, geometric shapes, place value manipulatives, fraction bars, etc.), drawings or diagrams (number lines, tally marks, tape diagrams, arrays, tables, graphs, etc.), paper and pencil, rulers and other measuring tools, scissors, tracing paper, grid paper, virtual manipulatives or other available technologies.

• Mathematically proficient elementary students choose tools that are relevant and useful to the problem at hand. These include such tools as are mentioned above as well as mathematical tools such as estimation or a particular strategy or algorithm. For example, in order to solve $3/5 - \frac{1}{2}$, a student might recognize that knowledge of equivalents of $\frac{1}{2}$ is an appropriate tool: since $\frac{1}{2}$ is equivalent to $2\frac{1}{2}$ fifths, the result is $\frac{1}{2}$ of a fifth or $\frac{1}{10}$.

• This practice is also related to looking for structure (MP.7), which often results in building mathematical tools that can then be used to solve problems.

6. Attend to precision.

Mathematically proficient students try to communicate precisely to others.• They try to use clear definitions in discussion with others and in their own reasoning.When making mathematical arguments about a solution, strategy, or conjecture (see MP.3), mathematically proficient elementary students learn to craft careful explanations that communicate their reasoning by referring specifically to each important mathematical element, describing the relationships among them, and connecting their words clearly to their representations. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately.• They are careful about specifying units of measure,• and labeling axes to clarify the correspondence with quantities in a problem.• They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

• Elementary students start by using everyday language to express their mathematical ideas, realizing that they need to select words with clarity and specificity rather than saying, for example, "it works" without explaining what "it" means. As they encounter the ambiguity of everyday terms, they come to appreciate, understand, and use mathematical vocabulary. Once young students become familiar with a mathematical idea or object, they are ready to learn more precise mathematical terms to describe it.

• For example, the equivalence of 8 and $5+3$ can be written both as $5+3=8$ and $8=5+3$. Similarly, the equivalence of $6+2$ and $5+3$ is expressed as $6+2=5+3$.

• When measuring, mathematically proficient elementary students use tools and strategies to minimize the introduction of error. From Kindergarten on, they count accurately, using strategies so that they include each object once and only once without losing track. They calculate accurately and efficiently and use clear and concise notation to record their work.

• In using representations, such as pictures, tables, graphs, or diagrams, they use appropriate labels to communicate the meaning of their representation.

7. Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure.• Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

• Mathematically proficient students at the elementary grades use structures such as place value, the properties of operations, other generalizations about the behavior of the operations (for example, the less you subtract, the greater the difference), and attributes of shapes to solve problems. In many cases, they have identified and described these structures through repeated reasoning (MP.8). For example, when younger students recognize that adding 1 results in the next counting number, they are identifying the basic structure of whole numbers. When older elementary students calculate 16×9, they might apply the structure of place value and the distributive property to find the product: $16 \times 9 = (10 + 6) \times 9 = (10 \times 9) + (6 \times 9)$. To determine the volume of a $3 \times 4 \times 5$ rectangular prism, students might see the structure of the prism as five layers of 3×4 arrays of cubes.

8. Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts.• Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal.• By paying attention to the calculation of slope as they repeatedly check whether points are on the line through $(1, 2)$ with slope 3, middle school students might abstract the equation $(y-2)/(x-1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x-1)(x+1)$, $(x-1)(x^2+x+1)$, and $(x-1)(x^3+x^2+x+1)$ might lead them to the general formula for the sum of a geometric series.• As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

• For example, younger students might notice that when tossing two-color counters to find combinations of a given number, they always get what they call "opposites"—when tossing 6 counters, they get 2 red, 4 yellow and 4 red, 2 yellow and when tossing 4 counters, they get 1 red, 3 yellow and 3 red, 1 yellow.

• Students in the middle elementary grades might notice a pattern in the change to the product when a factor is increased by 1: $5 \times 7 = 35$ and $5 \times 8 = 40$—the product changes by 5; $9 \times 4 = 36$ and $10 \times 4 = 40$—the product changes by 4. Students might then express this regularity by saying something like, "When you change one factor by 1, the product increases by the other factor."

• Mathematically proficient elementary students formulate conjectures about what they notice, for example, that when 1 is added to a factor, the product increases by the other factor; or that, whenever they toss counters, for each combination that comes up, its "opposite" can also come up. As students practice articulating their observations, they learn to communicate with greater precision (MP.6). As they explain why these generalizations must be true, they construct, critique, and compare arguments (MP.3).

K–5 elaborations of the standards for mathematical practice

The following elaborations of the practice standards integrate the commentary from the previous section into a single narrative describing a K–5 version of each standard for mathematical practice. This way of looking at the standards might be more useful than the previous secion in working with K–5 teachers.

1. Make sense of problems and persevere in solving them.

Mathematically proficient students at the elementary grades explain to themselves the meaning of a problem, look for entry points to begin work on the problem, and plan and choose a solution pathway. For example, young students might use concrete objects or pictures to show the actions of a problem, such as counting out and joining two sets to solve an addition problem. If students are not at first making sense of a problem or seeing a way to begin, they ask questions that will help them get started. As they work, they continually ask themselves, "Does this make sense?" When they find that their solution pathway does not make sense, they look for another pathway that does. They may consider simpler forms of the original problem; for example, to solve a problem involving multidigit numbers, they might first consider similar problems that involve multiples of ten or one hundred. Once they have a solution, they look back at the problem to determine if the solution is reasonable and accurate. They often check their answers to problems using a different method or approach.

Mathematically proficient students consider different representations of the problem and different solution pathways, both their own and those of other students, in order to identify and analyze correspondences among approaches. They can explain correspondences among physical models, pictures or diagrams, equations, verbal descriptions, tables, and graphs.

2. Reason abstractly and quantitatively.

Mathematically proficient students at the elementary grades make sense of quantities and their relationships in problem situations. They can contextualize quantities and operations by using images or stories. They interpret symbols as having meaning, not just as directions to carry out a procedure. Even as they manipulate the symbols, they can pause as needed to access the meaning of the numbers, the units, and the operations that the symbols represent. Mathematically proficient students know and flexibly use different properties of operations, numbers, and geometric objects.

Mathematically proficient students can contextualize an abstract problem by placing it in a context they then use to make sense of the mathematical ideas. For example, when a student sees the expression $40-26$, she might visualize this problem by thinking, if I have 26 marbles and Marie has 40, how many more do I need to have as many as Marie? Then, in that context, she thinks, 4 more will get me to a total of 30, and then 10 more will get me to 40, so the answer is 14. In this example, the student uses a context to think through a strategy for solving the problem, using the relationship between addition and subtraction and decomposing and recomposing the quantities. She then uses what she did in the context to identify the solution of the original abstract problem.

Mathematically proficient students can also make sense of a contextual problem and express the actions or events that are described in the problem using numbers and symbols. If they work with the symbols to solve the problem, they can then interpret their solution in terms of the context. For example, to find the area of the floor of a rectangular room that measures 10 m. by 12 m., a student might represent the problem as an equation, solve it mentally, and record the problem and solution as $10 \times 12 = 120$. He has decontextualized the problem. When he states at the end that the area of the room is 120 square meters, he has contextualized the answer in order to solve the original problem. Problems like this that begin with a context and are then represented with mathematical objects or symbols are also examples of modeling with mathematics (MP.4).

3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient students at the elementary grades construct mathematical arguments—that is, explain the reasoning underlying a strategy, solution, or conjecture—using concrete referents such as objects, drawings, diagrams, and actions. For example, in order to demonstrate what happens to the sum when the same amount is added to one addend and subtracted from another, students in the early grades might represent a story about children moving between two classrooms: the number of children in each classroom is an addend; the total number of children in the two classrooms is the sum. When some students move from one classroom to the other, the number of students in each classroom changes by that amount—one addend decreases by some amount and the other addend increases by that same amount—but the total number of students does not change. An older student might use an area representation to show why the distributive property holds. Arguments may also rely on definitions, previously established results, properties, or structures. For example, a student might argue that two different shapes have equal area because it has already been demonstrated that both shapes are half of the same rectangle. Students might also use counterexamples to argue that a conjecture is not true—for example, a rhombus is an example that shows that not all quadrilaterals with 4 equal sides are squares; or, multiplying by 1 shows that a product of two whole numbers is not always greater than each factor.

Mathematically proficient students present their arguments in the form of representations, actions on those representations, and explanations in words (oral or written). In the elementary grades, arguments are often a combination of all three. Some of their arguments apply to individual problems, but others are about conjectures based on regularities they have noticed across multiple problems (see MP.8, Look for and express regularity in repeated reasoning). As they articulate and justify generalizations, students consider to which mathematical objects (numbers or shapes, for example) their generalizations apply. For example, young students may believe a generalization about the behavior of addition applies to positive whole numbers less than 100 because those are the numbers with which they are currently familiar. As they expand their understanding of the number system, they may reexamine their conjecture for numbers in the hundreds and thousands. In upper elementary grades, students return to their conjectures and arguments about whole numbers to determine whether they apply to fractions and decimals. For example, students might make an argument based on an area representation of multiplication to show that the distributive property applies to problems involving fractions.

Mathematically proficient students can listen to or read the arguments of others, decide whether they make sense, ask useful questions to clarify or improve the arguments, and build on those arguments. They can communicate their arguments, compare them to others, and reconsider their own arguments in response to the critiques of others.

4. Model with mathematics.

When given a problem in a contextual situation, mathematically proficient students at the elementary grades can identify the mathematical elements of a situation and create a mathematical model that shows those mathematical elements and relationships among them. The mathematical model might be represented in one or more of the following ways: numbers and symbols, geometric figures, pictures or physical objects used to abstract the mathematical elements of the situation, or a mathematical diagram such as a number line, a table, or a graph, or students might use more than one of these to help them interpret the situation.

For example, when students are first studying an operation such as addition, they might arrange counters to solve problems such as this one: there are seven animals in the yard, some are dogs and some are cats, how many of each could there be? They are using the counters to model the mathematical elements of the contextual problem—that they can split a set of 7 into a set of 3 and a set of 4. When they learn how to write their actions with the counters in an equation, $4 + 3 = 7$, they are modeling the situation with numbers and symbols. Similarly, when students encounter situations such as sharing a pan of cornbread among 6 people, they might first show how to divide the cornbread into 6 equal pieces using a picture of a rectangle. The rectangle divided into 6 equal pieces is a model of the essential mathematical elements of the situation. When the students learn to write the name of each piece in relation to the whole pan as 1/6, they are now modeling the situation with mathematical notation.

Mathematically proficient students are able to identify important quantities in a contextual situation and use mathematical models to show the relationships of those quantities, particularly in multistep problems or problems involving more than one variable. For example, if there is a Penny Jar that starts with 3 pennies in the jar, and 4 pennies are added each day, students might use a table to model the relationship between number of days and number of pennies in the jar. They can then use the model to determine how many pennies are in the jar after 10 days, which in turn helps them model the situation with the expression, $4 \times 10 + 3$.

Mathematically proficient students use their model to analyze the relationships and draw conclusions. They interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

As students model situations with mathematics, they are choosing tools appropriately (MP.5). As they decontextualize the situation and represent it mathematically, they are also reasoning abstractly (MP.2).

5. Use appropriate tools strategically.

Mathematically proficient students at the elementary grades consider the tools that are available when solving a mathematical problem, whether in a real-world or mathematical context. These tools might include physical objects (cubes, geometric shapes, place value manipulatives, fraction bars, etc.), drawings or diagrams (number lines, tally marks, tape diagrams, arrays, tables, graphs, etc.), paper and pencil, rulers and other measuring tools, scissors, tracing paper, grid paper, virtual manipulatives or other available technologies. Proficient students are sufficiently familiar with tools appropriate for their grade and areas of content to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained from their use as well as their limitations.

Mathematically proficient students choose tools that are relevant and useful to the problem at hand. These include such tools as are mentioned above as well as mathematical tools such as estimation or a particular strategy or algorithm. For example, in order to solve $3/5 - \frac{1}{2}$, a student might recognize that knowledge of equivalents of $\frac{1}{2}$ is an appropriate tool: since $\frac{1}{2}$ is equivalent to $2\frac{1}{2}$ fifths, the result is $\frac{1}{2}$ of a fifth or $\frac{1}{10}$.

This practice is also related to looking for structure (MP.7), which often results in building mathematical tools that can then be used to solve problems.

6. Attend to precision.

Mathematically proficient students at the elementary grades communicate precisely to others. They start by using everyday language to express their mathematical ideas, realizing that they need to select words with clarity and specificity rather than saying, for example, "it works" without explaining what "it" means. As they encounter the ambiguity of everyday terms, they come to appreciate, understand, and use mathematical vocabulary. Once young students become familiar with a mathematical idea or object, they are ready to learn more precise mathematical terms to describe it. In using representations, such as pictures, tables, graphs, or diagrams, they use appropriate labels to communicate the meaning of their representation.

When making mathematical arguments about a solution, strategy, or conjecture (see MP.3), mathematically proficient students learn to craft careful explanations that communicate their reasoning by referring specifically to each important mathematical element, describing the relationships among them, and connecting their words clearly to their representations.

Elementary students learn to use mathematical symbols correctly and can describe the meaning of the symbols they use. In particular, they understand that the equal sign denotes that two quantities have the same value, and can use it flexibly to express equivalences. For example, the equivalence of 8 and $5 + 3$ can be written both as $5 + 3 = 8$ and $8 = 5 + 3$. Similarly, the equivalence of $6 + 2$ and $5 + 3$ is expressed as $6 + 2 = 5 + 3$.

When measuring, mathematically proficient students use tools and strategies to minimize the introduction of error. From Kindergarten on, they count accurately, using strategies so that they include each object once and only once without losing track. Mathematically proficient students calculate accurately and efficiently and use clear and concise notation to record their work.

7. Look for and make use of structure.

Mathematically proficient students at the elementary grades use structures such as place value, the properties of operations, other generalizations about the behavior of the operations (for example, the less you subtract, the greater the difference), and attributes of shapes to solve problems. In many cases, they have identified and described these structures through repeated reasoning (MP.8). For example, when younger students recognize that adding 1 results in the next counting number, they are identifying the basic structure of whole numbers. When older students calculate 16×9, they might apply the structure of place value and the distributive property to find the product: $16 \times 9 = (10+6) \times 9 = (10 \times 9) + (6 \times 9)$. To determine the volume of a $3 \times 4 \times 5$ rectangular prism, students might see the structure of the prism as five layers of 3×4 arrays of cubes.

8. Look for and express regularity in repeated reasoning.

Mathematically proficient students at the elementary grades look for regularities as they solve multiple related problems, then identify and describe these regularities. For example, students might notice a pattern in the change to the product when a factor is increased by 1: $5 \times 7 = 35$ and $5 \times 8 = 40$—the product changes by 5; $9 \times 4 = 36$ and $10 \times 4 = 40$—the product changes by 4. Students might then express this regularity by saying something like, "When you change one factor by 1, the product increases by the other factor." Younger students might notice that when tossing two-color counters to find combinations of a given number, they always get what they call "opposites"—when tossing 6 counters, they get 2 red, 4 yellow and 4 red, 2 yellow and when tossing 4 counters, they get 1 red, 3 yellow and 3 red, 1 yellow. Mathematically proficient students formulate conjectures about what they notice, for example, that when 1 is added to a factor, the product increases by the other factor; or that, whenever they toss counters, for each combination that comes up, its "opposite" can also come up. As students practice articulating their observations, they learn to communicate with greater precision (MP.6). As they explain why these generalizations must be true, they construct, critique, and compare arguments (MP.3).